CLASSICAL AND QUANTUM DYNAMICS OF NONCANONICALLY COUPLED OSCILLATORS AND LIE SUPERALGEBRAS

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Abstract. The classical and quantum dynamics of noncanonically coupled oscillators is investigated in its relation to Lie superalgebras. It is shown that the quantum dynamics admits a hidden (super)hamiltonian formulation and, hence, preserves the initial operator relations.

I. INTRODUCTION

It is well–known that classical and quantum dynamics of hamiltonian systems is often described by remarkable algebraic structures such as Lie algebras, their nonlinear generalizations and (quantum) deformations [1,2]. It seems that not less important objects govern a behaviour of the interacting hamiltonian systems and that they maybe unravelled in a certain way. There exist several forms of an interaction of hamiltonian systems: often it has a potential character, sometimes it is ruled by a deformation of the Poisson brackets; however, one of the most intriguing and mathematically less explored forms is a nonhamiltonian (noncanonical) interaction, which can not be described by deformations of the standard hamiltonian data (Poisson brackets and hamiltonians). Sometimes, such curious interaction is realized by the dependence (which is linear in the simplest cases [3] being nonlinear in general [4]) of the Poisson brackets of one hamiltonian system on the state of another [3]. The pair of noncanonically coupled oscillators is one of the simplest and the most crucial examples of the nonhamiltonian interaction [3]. The purpose of this paper is to describe the classical and quantum dynamics of noncanonically coupled oscillators in a general setting. For that we use the following algebraic structures (and their representations): (a) isotopic pairs (a particular linear case of general I–pairs of the note [4]), (b) anti–Jordan pairs, (c) anti–Lie triple systems, (d) Lie superalgebras. A brief description of relations between these algebraic structures is presented at the end of par.II.

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II. GENERAL DEFINITIONS

Definition 1 [3]**.** The pair (V_1, V_2) of linear spaces is called *an isotopic pair* iff there are defined two mappings $m_1 : V_2 \otimes \bigwedge^2 V_1 \mapsto V_1$ and $m_2 : V_1 \otimes \bigwedge^2 V_2 \mapsto V_2$ such that the mappings $(X,Y) \mapsto [X,Y]_A = m_1(A,X,Y)$ $(X,Y \in V_1,\ A \in V_2)$ and $(A,B) \mapsto [A,B]_X = m_2(X,A,B)$ $(A,B \in V_2,\ X \in V_1)$ obey the Jacobi identity for all values of a subscript parameter (such operations will be called *isocommutators* and the subscript parameters will be called *isotopic elements* or shortly *isotopies*) and are compatible to each other, i.e. the identities

$$\begin{aligned}[X,Y]_{[A,B]_Z} =&\tfrac{1}{2}([[X,Z]_A,Y]_B + [[X,Y]_A,Z]_B + [[Z,Y]_A,X]_B - \\ &[[X,Z]_B,Y]_A - [[X,Y]_B,Z]_A - [[Z,Y]_B,X]_A)\end{aligned}$$

and

$$\begin{aligned}[A,B]_{[X,Y]_C} =&\tfrac{1}{2}([[A,C]_X,B]_Y + [[A,B]_X,C]_Y + [[C,B]_X,A]_Y - \\ &[[A,C]_Y,B]_X - [[A,B]_Y,C]_X - [[C,B]_Y,A]_X)\end{aligned}$$

$(X,Y,Z \in V_1,\ A,B,C \in V_2)$ hold.

Let's discuss this defintion.

First, it may be considered as a result of an axiomatization of the following trivial construction: let $\mathcal{A}$ be an associative algebra (f.e. any matrix one) and V_1, V_2 be two linear subspaces in it such that V_1 is closed under the isocommutators $(X,Y) \mapsto [X,Y]_A = XAY - YAX$ with isotopic elements A from V_2, whereas V_2 is closed under the isocommutators $(A,B) \mapsto [A,B]_X = AXB - BXA$ with isotopic elements X from V_1.

Remark 1. Let H be a (finite dimensional) linear space. If $\mathcal{A}$ is a subspace of $\mathrm{End}(H)$ let's put $\mathcal{A}^{\mathfrak{i}} = \{X \in \mathrm{End}(H), \forall A \in \mathcal{A}, \forall B \in \mathcal{A}, AXB - BXA \in \mathcal{A}\}$. Then $\mathcal{A} \subseteq \mathcal{A}^{\mathfrak{ii}}$ and $(\mathcal{A}, \mathcal{A}^{\mathfrak{i}})$ is an isotopic pair.

It is rather interesting to unravel the most general setting for such construction. Namely, let $[\cdot,\cdot]_\alpha$ $(\alpha \in \mathfrak{A})$ be a linear family of compatible Lie brackets on V. When the bracket $[\cdot,\cdot]_{\alpha \underset{Z}{\Diamond} \beta}$ $(Z \in V)$ defined on V as

$$[X,Y]_{\alpha \underset{Z}{\Diamond} \beta} = \tfrac{1}{2}([[X,Z]_A,Y]_B + [[X,Y]_A,Z]_B + [[Z,Y]_A,X]_B - (\alpha \longleftrightarrow \beta))$$

is a Lie bracket compatible with brackets $[\cdot,\cdot]_\alpha$ and $[\cdot,\cdot]_\beta$ $(\alpha, \beta \in \mathfrak{A})$?

Certainly, the $\mathfrak{g}$–equivariant case is the most important one but, unfortunately, I do not know any answer on this simple question. The existence of $\Diamond$–operation on the space of compatible Lie brackets depends on the correctness of the following *conjecture*: let $[\cdot,\cdot]_\alpha$ $(\alpha \in \mathfrak{A})$ be a linear family of compatible Lie brackets on the linear space V, then there exist a linear space H and two linear mappings $T \in \mathrm{Hom}(V;\mathrm{End}(H))$ and $Q \in \mathrm{Hom}(\mathfrak{A};\mathrm{End}(H))$ such that $T([X,Y]_\alpha) = T(X)Q(\alpha)T(Y) - T(Y)Q(\alpha)T(X)$.

Remark 2. Let $\mathcal{A} \subseteq \mathrm{End}(H)$, $\dim \mathcal{A} = n$, Lie_n be the space of all Lie algebras of dimension n. Then there exists a natural mapping $\mathcal{L} : \mathcal{A}^{\mathfrak{i}} \mapsto \mathrm{Lie}_n$. It should be mentioned that $\mathrm{card}\,\mathcal{L}(\mathcal{A}^{\mathfrak{i}})$ may be not equal to 1 so $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ are not the same in general for different X and Y. It means that the isocommutators $[\cdot,\cdot]_X$ and $[\cdot,\cdot]_Y$ determine structures of nonisomorphic Lie algebras on the space $\mathcal{A}$ in general (though they may be isomorphic in particular).

Second, one may compare def.1 with the definition of "anti–Jordan pairs" [5]. Namely,

Definition 2 (cf.[6])**.** The pair (V_1, V_2) of linear spaces is called *an anti–Jordan pair* iff there are defined two mappings $m_1 : V_2 \otimes \bigwedge^2 V_1 \mapsto V_1$ and $m_2 : V_1 \otimes \bigwedge^2 V_2 \mapsto V_2$ such that the mappings $(X, Y) \mapsto [X, Y]_A = m_1(A, X, Y)$ $(X, Y \in V_1,\ A \in V_2)$ and $(A, B) \mapsto [A, B]_X = m_2(X, A, B)$ $(A, B \in V_2,\ X \in V_1)$ are compatible to each other in the following manner

$$[X, Y]_{[A,B]_Z} = [[X, Z]_A, Y]_B + [[Z, Y]_A, X]_B - [[X, Y]_B, Z]_A$$

and

$$[A, B]_{[X,Y]_C} = [[A, C]_X, B]_Y + [[C, B]_X, A]_Y - [[A, B]_Y, C]_X$$

$(X, Y, Z \in V_1,\ A, B, C \in V_2)$ hold.

It can be easily verified that isotopic pairs are always anti–Jordan pairs (to obtain it one should use the Jacobi identity linearized by subscript parameters), and that the anti–Jordan pairs with a multiplication, obeying Jacobi identity if a subscript parameter is fixed in any way, are just the isotopic pairs. So the isotopic pairs may be considered as a particular case of the anti–Jordan pairs. Note that there exist examples of anti–Jordan pairs, which are not isotopic ones [5].

Anti–Jordan pairs are closely related to the (polarized) anti–Lie triple systems and Lie superalgebras [5] (cf. also [7]). Namely,

Definition 3. The ternary algebra V with product $[xyz]$ is called *an anti–Lie triple system* if

(1) $[xyz] = [xzy]$,
(2) $[xyz] + [zxy] + [yzx] = 0$,
(3) $[[xyz]uv] = [[xuv]yz] + [x[yvu]z] + [xy[zuv]]$.

An anti–Lie triple system V is *polarized* iff $V = V_1 \oplus V_2$ and $[xyz] = 0$ for $y, z \in V_1$ or $y, z \in V_2$.

If V is an anti–Lie triple system let's put $R_{yz} \in \mathrm{End}(V) : R_{yz}x = [xyz]$. The operators R_{yz} are closed under commutators so that $\mathfrak{g}_0(V) = \mathrm{span}(R_{yz}; y, z \in V)$ is a Lie algebra. The space $\mathfrak{g}_0(V) \oplus V$ possesses a natural structure of a Lie superalgebra [8] with the even part $\mathfrak{g}_0(V)$ and the odd part V [5]. It will be denoted by $\mathfrak{g}(V)$. Polarized anti–Lie triple systems $V = V_1 \oplus V_2$ produce polarized Lie superalgebras $\mathfrak{g}(V) = \mathfrak{g}_0(V) \oplus (V_1 \oplus V_2)$ such that $[V_i, V_i]_+ = 0$, $[\mathfrak{g}(V), V_i]_- \subseteq V_i$ (it should be marked that there is sometimes asserted that $V_2 \simeq V_1^*$ as $\mathfrak{g}_0(V)$–modules, however, we shall not do it in general).

An arbitrary anti–Jordan pair (so an isotopic pair, in particular) has a structure of a polarized anti–Lie triple system. Namely, one should put $[xyz] = [z, x]_y$ (iff z belongs to the same space V_i as x) and $[y, x]_z$ (iff y belongs to the same space V_i as x).

Remark 3. Let's summarize the relations between the concepts of "isotopic pair", "anti–Jordan pair", "polarized anti–Lie triple system" and "polarized Lie superalgebra" once more.

(1) Each isotopic pair is an anti–Jordan pair, though there exist anti–Jordan pairs, which are not isotopic. It means that isotopic pairs form a proper subclass of the class of anti–Jordan pairs.

(2) Categories of anti–Jordan pairs and polarized anti–Lie triple systems are equivalent. It means that each anti–Jordan pair defines a polarized anti–Lie triple system and vice versa.
(3) Categories of polarized anti–Lie triple systems and polarized Lie superalgebras are equivalent. On the other hand polarized anti–Lie triple systems and polarized Lie superalgebras are particular cases of anti–Lie triple systems and Lie superalgebras respectively and the marked equivalency of categories is a particular case of the equivalency of categories of anti–Lie triple systems and Lie superalgebras.
(4) As a consequence of (2) and (3) categories of anti–Jordan pairs and polarized Lie superalgebras are equivalent.
(5) As a consequence of (1) and (4) each isotopic pair defines a polarized Lie superalgebra but not vice versa.

An illustrative example to the construction of a Lie superalgebra by an isotopic pair is convenient. *Example*: let H_1 and H_2 be two linear spaces, $(\mathrm{Hom}(H_1, H_2); \mathrm{Hom}(H_2, H_1))$ is an isotopic pair, the corresponding Lie superalgebra is isomorphic to $\mathfrak{gl}(n|m)$, $n = \dim H_1$, $m = \dim H_2$.

Note that the isocommutators in an isotopic pair (V_1, V_2) define families of Poisson brackets $\{\cdot,\cdot\}_A$ and $\{\cdot,\cdot\}_X$ ($A \in V_2$, $X \in V_1$) in the spaces $S^{\cdot}(V_1)$ and $S^{\cdot}(V_2)$, respectively.

Definition 4 (cf.[3])**.** Let's consider two elements $\mathcal{H}_1$ and $\mathcal{H}_2$ (*"hamiltonians"*) in $S^{\cdot}(V_1)$ and $S^{\cdot}(V_2)$, respectively. The equations

$$\dot{X}_t = \{\mathcal{H}_1, X_t\}_{A_t}, \qquad \dot{A}_t = \{\mathcal{H}_2, A_t\}_{X_t},$$

where $X_t \in V_1$ and $A_t \in V_2$ are called *the (nonlinear) dynamical equations associated with the isotopic pair* (V_1, V_2) *and "hamiltonians"* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ (it should be marked that "hamiltonians" are not even integrals of motion in a general situation).

It should be mentioned that the dynamical equations associated with isotopic pairs are a particular case of such equations associated with general I–pairs [4].

III. ISOTOPIC PAIR OF NONCANONICALLY COUPLED OSCILLATORS: ALGEBRAIC ASPECTS

Let's now consider the isotopic pairs of noncanonically coupled oscillators [3,6]. The space V_1 is spanned by the elements p, q and r and the space V_2 is spanned by the elements a, b and c. The isocommutators have the form

$$\begin{array}{lll}
[p,q]_a = 2\varepsilon_1 q & [p,q]_b = 2\varepsilon_1 p & [p,q]_c = \varepsilon_3 r \\
[p,r]_a = \varepsilon_2 r & [p,r]_b = 0 & [p,r]_c = 0 \\
[q,r]_a = 0 & [q,r]_b = -\varepsilon_2 r & [q,r]_c = 0
\end{array}$$

$$\begin{array}{lll}
[a,b]_p = 2\tilde{\varepsilon}_1 b & [a,b]_q = 2\tilde{\varepsilon}_1 a & [a,b]_r = \tilde{\varepsilon}_3 c \\
[a,c]_p = \tilde{\varepsilon}_2 c & [a,c]_q = 0 & [b,c]_r = 0 \\
[b,c]_p = 0 & [b,c]_q = -\tilde{\varepsilon}_2 c & [a,c]_r = 0
\end{array}$$

where

$$\begin{cases} \varepsilon_1 + \tilde{\varepsilon}_1 = 0 \\ \varepsilon_2 - \tilde{\varepsilon}_2 = \varepsilon_1 - \tilde{\varepsilon}_1 \\ \varepsilon_3\tilde{\varepsilon}_3 - \varepsilon_2\tilde{\varepsilon}_2 = 0 \end{cases}$$

The corresponding Lie algebra $\mathfrak{g}_0(V_1 \oplus V_2)$ is spanned (for generic ε_i, $\tilde{\varepsilon}_i$) by 6 operators $R_{p,a}$, $R_{p,b}$, $R_{q,a}$, $R_{q,b}$, $R_{r,b} = \frac{\varepsilon_2}{\varepsilon_3}R_{p,c}$, $R_{r,a} = \frac{\varepsilon_2}{\varepsilon_3}R_{q,c}$, which have the form

$$R_{p,a} = \begin{pmatrix} 2\varepsilon_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varepsilon_2 \end{pmatrix}, \quad R_{p,b} = \begin{pmatrix} 0 & 0 & 0 \\ 2\varepsilon_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$R_{q,a} = \begin{pmatrix} 0 & -2\varepsilon_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad R_{q,b} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2\varepsilon_1 & 0 \\ 0 & 0 & -\varepsilon_2 \end{pmatrix},$$
$$R_{p,c} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varepsilon_3 & 0 & 0 \end{pmatrix}, \quad R_{q,c} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\varepsilon_3 & 0 \end{pmatrix}$$

in the basis (q, p, r) and the form

$$R_{p,a} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\tilde{\varepsilon}_1 & 0 \\ 0 & 0 & \tilde{\varepsilon}_2 \end{pmatrix}, \quad R_{p,b} = \begin{pmatrix} 0 & 0 & 0 \\ -2\tilde{\varepsilon}_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$R_{q,a} = \begin{pmatrix} 0 & 2\tilde{\varepsilon}_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad R_{q,b} = \begin{pmatrix} -2\tilde{\varepsilon}_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tilde{\varepsilon}_2 \end{pmatrix},$$
$$R_{p,c} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\tilde{\varepsilon}_2 & 0 & 0 \end{pmatrix}, \quad R_{q,c} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \tilde{\varepsilon}_2 & 0 \end{pmatrix}$$

in the basis (a, b, c).

The Lie superalgebra $\mathfrak{g}(V_1 \oplus V_2)$ has a (super)dimension $(6|6)$ and is generated by $R_{p,a}$, $R_{p,b}$, $R_{q,a}$, $R_{q,b}$, $R_{p,c}$, $R_{q,c}$, p, q, r, a, b, c with (super)commutation relations

$$\begin{gathered} [q,p]_+ = [q,r]_+ = [p,r]_+ = [a,b]_+ = [a,c]_+ = [b,c]_+ = [r,c]_+ = 0, \\ [p,a]_+ = R_{p,a}, \ [q,a]_+ = R_{q,a}, \ [p,b]_+ = R_{p,b}, \\ [q,b]_+ = R_{q,b}, \ [p,c]_+ = R_{p,c}, \ [q,c]_+ = R_{q,c}, \\ [r,a]_+ = \tfrac{\varepsilon_2}{\varepsilon_3}R_{q,c}, \ [r,b]_+ = \tfrac{\varepsilon_2}{\varepsilon_3}R_{p,c}; \end{gathered}$$

$$
\begin{gathered}
[R_{p,a}, q]_- = 2\varepsilon_1 q,\ [R_{p,a}, p]_- = 0,\ [R_{p,a}, r]_- = \varepsilon_2 r,\\
[R_{q,a}, q]_- = 0,\ [R_{q,a}, p]_- = -2\varepsilon_1 q,\ [R_{q,a}, r]_- = 0,\\
[R_{p,b}, q]_- = 2\varepsilon_1 p,\ [R_{p,b}, p]_- = 0,\ [R_{p,b}, r]_- = 0,\\
[R_{q,b}, q]_- = 0,\ [R_{q,b}, p]_- = -2\varepsilon_1 p,\ [R_{q,b}, r]_- = -\varepsilon_2 r,\\
[R_{p,c}, q]_- = \varepsilon_3 r,\ [R_{p,c}, p]_- = 0,\ [R_{p,c}, r]_- = 0,\\
[R_{q,c}, q]_- = 0,\ [R_{q,c}, p]_- = -\varepsilon_3 r,\ [R_{q,c}, r]_- = 0,\\
[R_{p,a}, a]_- = 0,\ [R_{p,a}, b]_- = 2\tilde{\varepsilon}_1 b,\ [R_{p,a}, c]_- = \tilde{\varepsilon}_2 c,\\
[R_{q,a}, a]_- = 0,\ [R_{q,a}, b]_- = 2\tilde{\varepsilon}_1 a,\ [R_{q,a}, c]_- = 0,\\
[R_{p,b}, a]_- = -2\tilde{\varepsilon}_1 b,\ [R_{p,b}, b]_- = 0,\ [R_{p,b}, c]_- = 0,\\
[R_{q,b}, a]_- = -2\tilde{\varepsilon}_1 a,\ [R_{q,b}, b]_- = 0,\ [R_{q,b}, c]_- = -\tilde{\varepsilon}_2 c,\\
[R_{p,c}, a]_- = -\tilde{\varepsilon}_2 c,\ [R_{p,c}, b]_- = 0,\ [R_{p,c}, c]_- = 0,\\
[R_{q,c}, a]_- = 0,\ [R_{q,c}, b]_- = \tilde{\varepsilon}_2 c,\ [R_{q,c}, c]_- = 0;
\end{gathered}
$$

$$
\begin{gathered}
[R_{p,a}, R_{p,b}]_- = -2\varepsilon_1 R_{p,b},\ [R_{p,a}, R_{q,a}]_- = 2\varepsilon_1 R_{q,a},\ [R_{p,a}, R_{p,b}]_- = 0,\\
[R_{p,a}, R_{p,c}]_- = \tilde{\varepsilon}_2 R_{p,c},\ [R_{p,a}, R_{q,c}]_- = \varepsilon_2 R_{q,c},\ [R_{p,b}, R_{q,a}]_- = 2\varepsilon_1 (R_{q,b} + R_{p,a}),\\
[R_{p,b}, R_{q,b}]_- = 2\varepsilon_1 R_{p,b},\ [R_{p,b}, R_{p,c}]_- = 0,\ [R_{p,b}, R_{q,c}]_- = 2\varepsilon_1 R_{p,c},\\
[R_{q,a}, R_{q,b}]_- = -2\varepsilon_1 R_{q,a},\ [R_{q,a}, R_{p,c}]_- = -2\varepsilon_1 R_{p,c},\ [R_{q,a}, R_{q,c}]_- = 0,\\
[R_{q,b}, R_{p,c}]_- = -\varepsilon_2 R_{p,c},\ [R_{q,b}, R_{q,c}]_- = -\tilde{\varepsilon}_2 R_{q,c},\ [R_{p,c}, R_{q,c}]_- = 0.
\end{gathered}
$$

The even part of the Lie superalgebra $\mathfrak{g}(V_1 \oplus V_2)$ is isomorphic to the semidirect sum of $\mathfrak{gl}(2, \mathbb{C})$ and $\mathbb{C}^2$. On the other hand $\mathfrak{g}(V_1 \oplus V_2)$ may be considered as a semidirect product of the Lie superalgebra $\mathfrak{sl}(2|1, \mathbb{C})$ generated by $R_{p,a}$, $R_{p,b}$, $R_{q,a}$, $R_{q,b}$, p, q, a, b and the $(2|2)$–dimensional vector superspace $V^{2|2}$ generated by $R_{p,c}$, $R_{q,c}$, r, c.

IV. ISOTOPIC PAIR OF NONCANONICALLY COUPLED OSCILLATORS: CLASSICAL DYNAMICS

The dynamical equations with "hamiltonians" $\mathcal{H}_1 = P^2 + Q^2$ and $\mathcal{H}_2 = A^2 + B^2$ have the form

$$
\begin{cases}
\dot{P} = -\,4\varepsilon_1 (Q^2 A + PQB) - 2\varepsilon_3 RQC\\
\dot{Q} = 4\varepsilon_1 (PQA + P^2 B) + 2\varepsilon_3 RPC\\
\dot{R} = 2\varepsilon_2 (PRA - QRB)
\end{cases}
$$

$$
\begin{cases}
\dot{A} = -\,4\tilde{\varepsilon}_1 (B^2 P + ABQ) - 2\tilde{\varepsilon}_3 CBR\\
\dot{B} = 4\tilde{\varepsilon}_1 (ABP + A^2 Q) + 2\tilde{\varepsilon}_3 CAR\\
\dot{C} = 2\tilde{\varepsilon}_2 (ACP - BCQ)
\end{cases}
$$

Note that "hamiltonians" $\mathcal{H}_1 = \mathcal{I}_1^2$ and $\mathcal{H}_2 = \mathcal{I}_2^2$ are integrals of motion here, so it is rather convenient to put $P = \mathcal{I}_1 \cos\varphi$, $Q = \mathcal{I}_1 \sin\varphi$, $A = \mathcal{I}_2 \cos\psi$, $B = \mathcal{I}_2 \sin\psi$. Then

$$
\begin{cases}
\dot{\varphi} = -\,2\varepsilon_3 RC - 4\varepsilon_1 \mathcal{I}_1 \mathcal{I}_2 \sin(\varphi + \psi)\\
\dot{\psi} = -\,2\tilde{\varepsilon}_3 RC - 4\tilde{\varepsilon}_1 \mathcal{I}_1 \mathcal{I}_2 \sin(\varphi + \psi)
\end{cases}
$$

$$\begin{cases} \dot{R} = & 2\varepsilon_2 \cos(\varphi+\psi) R \\ \dot{C} = & 2\tilde{\varepsilon}_2 \cos(\varphi+\psi) C \end{cases}$$

Let's introduce $\vartheta = \varphi + \psi$, $\chi = \varepsilon_3\psi - \tilde{\varepsilon}_3\varphi$ and mark that $\varepsilon_1 + \tilde{\varepsilon}_1 = 0$, then

$$\begin{cases} \dot{\vartheta} = & -2(\varepsilon_3 + \tilde{\varepsilon}_3) RC \\ \dot{\chi} = & 4\varepsilon_1 \mathcal{I}_1 \mathcal{I}_2 (\varepsilon_3 - \tilde{\varepsilon}_3) \sin\vartheta \end{cases}$$

Also

$$(RC)^{\cdot} = 2(\varepsilon_2 + \tilde{\varepsilon}_2) \cos\vartheta (RC),$$

therefore,

$$(RC)'_{\vartheta} = -\frac{\varepsilon_2 + \tilde{\varepsilon}_2}{\varepsilon_3 + \tilde{\varepsilon}_3} \cos\vartheta$$

and

$$RC = \mathcal{L} - \frac{\varepsilon_2 + \tilde{\varepsilon}_2}{\varepsilon_3 + \tilde{\varepsilon}_3} \mathcal{I}_1 \mathcal{I}_2 \sin\vartheta,$$

whereas

$$\dot{\vartheta} = -2\mathcal{L}(\varepsilon_3 + \tilde{\varepsilon}_3) - 2\mathcal{I}_1 \mathcal{I}_2 (\varepsilon_2 + \tilde{\varepsilon}_2) \sin\vartheta.$$

Here $\mathcal{L} = RC + \frac{\varepsilon_2 + \tilde{\varepsilon}_2}{\varepsilon_3 + \tilde{\varepsilon}_3}(QA + PB)$ is an integral of motion. Note that

$$(R^{\tilde{\varepsilon}_2} C^{-\varepsilon_2})^{\cdot} = 0$$

so it is convenient to put

$$\Lambda = R^{\frac{\tilde{\varepsilon}_2}{\varepsilon_2 + \tilde{\varepsilon}_2}} C^{\frac{\varepsilon_2}{\varepsilon_2 + \tilde{\varepsilon}_2}}.$$

Then

$$\begin{cases} R = & \Lambda \left(\mathcal{L} - \frac{\varepsilon_2 + \tilde{\varepsilon}_2}{\varepsilon_3 + \tilde{\varepsilon}_3} \mathcal{I}_1 \mathcal{I}_2 \sin\vartheta\right)^{\frac{\varepsilon_2}{\varepsilon_2 + \tilde{\varepsilon}_2}} \\ C = & \frac{1}{\Lambda} \left(\mathcal{L} - \frac{\varepsilon_2 + \tilde{\varepsilon}_2}{\varepsilon_3 + \tilde{\varepsilon}_3} \mathcal{I}_1 \mathcal{I}_2 \sin\vartheta\right)^{\frac{\tilde{\varepsilon}_2}{\varepsilon_2 + \tilde{\varepsilon}_2}} \end{cases}$$

$\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{L}$ and Λ form a complete set of integrals of motion for generic values of ε_i, $\tilde{\varepsilon}_i$.

Let's also denote

$$\begin{aligned} \xi &= (\varepsilon_2 + \tilde{\varepsilon}_2)\chi + 2\varepsilon_1(\varepsilon_3 - \tilde{\varepsilon}_3)\vartheta \\ &= [(\varepsilon_2 + \tilde{\varepsilon}_2)\varepsilon_3 + 2\varepsilon_1(\varepsilon_3 - \tilde{\varepsilon}_3)]\psi - [(\varepsilon_2 + \tilde{\varepsilon}_2)\tilde{\varepsilon}_3 + 2\tilde{\varepsilon}_1(\varepsilon_3 - \tilde{\varepsilon}_3)]\varphi, \end{aligned}$$

then

$$\xi = 4\mathcal{L}(\tilde{\varepsilon}_3^2 - \varepsilon_3^2)\varepsilon_1 t + \xi_0.$$

The obtained results certainly generalize results of [3]. It is also possible to consider a hybrid coupling for the isotopic pairs of noncanonically coupled oscillators (cf.[3]).

V. ISOTOPIC PAIR OF NONCANONICALLY COUPLED OSCILLATORS: REPRESENTATION THEORY AND QUANTUM DYNAMICS

Definition 5. *A representation of the isotopic pair* (V_1, V_2) *in the linear space* W is a pair (T_1, T_2) of mappings $T_i : V_i \mapsto \mathrm{End}(W)$ such that

$$\begin{aligned} T_1([X,Y]_A) =& T_1(X)T_2(A)T_1(Y) - T_1(Y)T_2(A)T_1(X), \\ T_2([A,B]_X) =& T_2(A)T_1(X)T_2(B) - T_2(B)T_1(X)T_2(A), \end{aligned}$$

where $X, Y \in V_1,\ A, B \in V_2$ [6]. A representation of the isotopic pair (V_1, V_2) in the linear space W is called *nilpotent* if

$$\begin{cases} (\forall X_i \in V_1)\ T_1(X_1)T_1(X_2) = 0, \\ (\forall A_i \in V_2)\ T_2(A_1)T_2(A_2) = 0. \end{cases}$$

A representation of the isotopic pair (V_1, V_2) in the linear space W is called *split* [6] iff $W = W_1 \oplus W_2$ and

$$\begin{cases} (\forall X \in V_1)\ T_1(X)|_{W_2} = 0,\ T_1(X) : W_1 \mapsto W_2, \\ (\forall A \in V_2)\ T_2(A)|_{W_1} = 0,\ T_2(A) : W_2 \mapsto W_1. \end{cases}$$

Otherwords, operators $T(X)$ and $T(A)$ have the form $\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$, respectively. Each split representation is nilpotent.

Not that a nilpotent representation (but not an arbitrary one) of an isotopic pair (V_1, V_2) defines a representation T of the corresponding anti–Lie triple system and Lie superalgebra $\mathfrak{g}(V_1 \oplus V_2)$ (or its central extension $\hat{\mathfrak{g}}(V_1 \oplus V_2)$). It should be stressed once more that a representation of the polarized Lie superalgebra, constructed by an isotopic pair, may not define a representation of the least. If the representation of the isotopic pair (V_1, V_2) is split then the representation of the Lie superalgebra $\mathfrak{g}(V_1 \oplus V_2)$ always have a special "polarized" form: $W = W_1 \oplus W_2$, $T(V_1) : W_1 \mapsto W_2$, $T(V_2) : W_2 \mapsto W_1$, $\mathfrak{g}_0(V_1 \oplus V_2) : W_i \mapsto W_i$.

Note that each representation (T_1, T_2) of the isotopic pair (V_1, V_2) in the space W defines a split representation (T_1^s, T_2^s) of the same pair in the space $W_1 \oplus W_2$ $(W_i \simeq W)$:

$$(\forall X \in V_1)\ T_1^s(X) = \begin{pmatrix} 0 & 0 \\ T_1(X) & 0 \end{pmatrix}, \quad (\forall A \in V_2)\ T_2^s(A) = \begin{pmatrix} 0 & T_2(A) \\ 0 & 0 \end{pmatrix}.$$

To construct a split representation of an isotopic pair (V_1, V_2) one may start from two arbitrary $\mathfrak{g}_0(V_1 \oplus V_2)$–modules W_1 and W_2, to consider suitable $\mathfrak{g}_0(V_1 \oplus V_2)$–tensor operators [9] from W_1 to W_2 and vice versa as candidates for $T(V_1)$ and $T(V_2)$, respectively, and then to check the validity of anticommutation relations between the tensor operators. In such approach elements of the isotopic pair (V_1, V_2) are realized as hidden symmetries with respect to $\mathfrak{g}_0(V_1 \oplus V_2)$ (cf.[10,11]).

Let's now describe the quantum dynamics of noncanonically coupled oscillators

Mark that the classical dynamics of noncanonically coupled oscillators was not somehow naturally related to Lie superalgebras, the situation in the quantum case is rather different. Namely, the formal quantum dynamical equations have the form

$$\begin{cases} \frac{d}{dt}\hat{P}_t = -2\varepsilon_1(\hat{P}_t\hat{B}_t\hat{Q}_t + \hat{Q}_t\hat{B}_t\hat{P}_t + 2\hat{Q}_t\hat{A}_t\hat{Q}_t) - \varepsilon_3(\hat{R}_t\hat{C}_t\hat{Q}_t + \hat{Q}_t\hat{C}_t\hat{R}_t) \\ \frac{d}{dt}\hat{Q}_t = 2\varepsilon_1(\hat{P}_t\hat{A}_t\hat{Q}_t + \hat{Q}_t\hat{A}_t\hat{P}_t + 2\hat{P}_t\hat{B}_t\hat{P}_t) + \varepsilon_3(\hat{R}_t\hat{C}_t\hat{P}_t + \hat{P}_t\hat{C}_t\hat{R}_t) \\ \frac{d}{dt}\hat{R}_t = \varepsilon_2(\hat{P}_t\hat{A}_t\hat{R}_t + \hat{R}_t\hat{A}_t\hat{P}_t - \hat{Q}_t\hat{B}_t\hat{R}_t - \hat{R}_t\hat{B}_t\hat{Q}_t) \end{cases}$$

$$\begin{cases} \frac{d}{dt}\hat{A}_t = -2\tilde{\varepsilon}_1(\hat{A}_t\hat{Q}_t\hat{B}_t + \hat{B}_t\hat{Q}_t\hat{A}_t + 2\hat{B}_t\hat{P}_t\hat{B}_t) - \tilde{\varepsilon}_3(\hat{C}_t\hat{R}_t\hat{B}_t + \hat{B}_t\hat{R}_t\hat{C}_t) \\ \frac{d}{dt}\hat{B}_t = 2\tilde{\varepsilon}_1(\hat{A}_t\hat{P}_t\hat{B}_t + \hat{B}_t\hat{P}_t\hat{A}_t + 2\hat{A}_t\hat{Q}_t\hat{A}_t) + \tilde{\varepsilon}_3(\hat{C}_t\hat{R}_t\hat{A}_t + \hat{A}_t\hat{R}_t\hat{C}_t) \\ \frac{d}{dt}\hat{C}_t = \tilde{\varepsilon}_2(\hat{A}_t\hat{P}_t\hat{C}_t + \hat{C}_t\hat{P}_t\hat{A}_t - \hat{B}_t\hat{Q}_t\hat{C}_t - \hat{C}_t\hat{Q}_t\hat{B}_t) \end{cases}$$

The dynamics is considered in arbitrary representation of the isotopic pair of noncanonically coupled oscillators. Let's consider such dynamics in the corresponding split representation. First of all renormalize c and r so that $R_{p,c} = R_{b,r}$ and $R_{q,c} = R_{a,r}$. Then the following proposition holds.

Proposition 1. *Equations of quantum dynamics of noncanonically coupled oscillators are a reduction of formal super Heisenberg equations*

$$\frac{d}{dt}\hat{F}_t = [\hat{H}_{\text{hidden}}, \hat{F}_t]$$

in $\mathcal{U}(\mathfrak{g}(V_1 \oplus V_2))$ *with quadratic quantum hamiltonian*

$$\hat{H}_{\text{hidden}} = \hat{R}^2_{q,a} + \hat{R}^2_{p,b} + \hat{R}^2_{q,b} + \hat{R}^2_{p,a} + \hat{R}^2_{p,c} + \hat{R}^2_{q,c}$$

So quantum dynamics of noncanonically coupled oscillators admits a hidden super–hamiltonian formulation in terms of Lie superalgebra $\mathfrak{g}(V_1 \oplus V_2)$.

Corollary. *The quantum dynamics preserves the initial operator relations:*

$$\hat{P}_t\hat{A}_t\hat{Q}_t - \hat{Q}_t\hat{A}_t\hat{P}_t = 2\varepsilon_1\hat{Q}_t,\ \hat{P}_t\hat{A}_t\hat{R}_t - \hat{R}_t\hat{A}_t\hat{P}_t = \varepsilon_2\hat{R}_t,\ \hat{Q}_t\hat{A}_t\hat{R}_t - \hat{R}_t\hat{A}_t\hat{Q}_t = 0,$$
$$\hat{P}_t\hat{B}_t\hat{Q}_t - \hat{Q}_t\hat{B}_t\hat{P}_t = 2\varepsilon_1\hat{P}_t,\ \hat{P}_t\hat{B}_t\hat{R}_t - \hat{R}_t\hat{B}_t\hat{P}_t = 0,\ \hat{Q}_t\hat{B}_t\hat{R}_t - \hat{R}_t\hat{B}_t\hat{Q}_t = -2\varepsilon_2\hat{R}_t,$$
$$\hat{P}_t\hat{C}_t\hat{Q}_t - \hat{Q}_t\hat{C}_t\hat{P}_t = \varepsilon_3\hat{R}_t,\ \hat{P}_t\hat{C}_t\hat{R}_t - \hat{R}_t\hat{C}_t\hat{P}_t = 0,\ \hat{Q}_t\hat{C}_t\hat{R}_t - \hat{R}_t\hat{C}_t\hat{Q}_t = 0,$$

$$\hat{A}_t\hat{P}_t\hat{B}_t - \hat{B}_t\hat{P}_t\hat{A}_t = 2\tilde{\varepsilon}_1\hat{B}_t,\ \hat{A}_t\hat{P}_t\hat{C}_t - \hat{C}_t\hat{P}_t\hat{A}_t = \tilde{\varepsilon}_2\hat{C}_t,\ \hat{B}_t\hat{P}_t\hat{C}_t - \hat{C}_t\hat{P}_t\hat{B}_t = 0,$$
$$\hat{A}_t\hat{Q}_t\hat{B}_t - \hat{B}_t\hat{Q}_t\hat{A}_t = 2\tilde{\varepsilon}_1\hat{A}_t,\ \hat{A}_t\hat{Q}_t\hat{C}_t - \hat{C}_t\hat{Q}_t\hat{A}_t = 0,\ \hat{B}_t\hat{Q}_t\hat{C}_t - \hat{C}_t\hat{Q}_t\hat{B}_t = -\tilde{\varepsilon}_2\hat{C}_t,$$
$$\hat{A}_t\hat{R}_t\hat{B}_t - \hat{B}_t\hat{R}_t\hat{A}_t = \tilde{\varepsilon}_3\hat{C}_t,\ \hat{B}_t\hat{R}_t\hat{C}_t - \hat{C}_t\hat{R}_t\hat{B}_t = 0,\ \hat{A}_t\hat{R}_t\hat{C}_t - \hat{C}_t\hat{R}_t\hat{A}_t = 0.$$

VI. REMARKS

Remark 4. Note that the dynamics of the classical Poisson brackets is not defined *as such*, because the subscript parameters are considered to be only linear functions. Hence, to define their conserved dynamics it is necessary to extend them correctly to arbitrary subscript parameters.

Remark 5. It is interesting to consider a quantum version of hybrid couplings [3].

Remark 6. Though the representation theory of isotopic pairs may be imbed into the representation theory of (polarized) Lie superalgebras, sometimes it is rather reasonable to avoid it in view of the dimension growing (f.e. for isotopic pairs of symmetric and skew–symmetric matrices [2,5] or for geometric isotopic pairs [5]).

VII. CONCLUSIONS

Thus, the classical and quantum dynamics of noncanonically coupled oscillators is investigated. The crucial role of Lie superalgebras is explicated. It is shown that quantum dynamics preserves the initial operator relations.

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APPENDIX A: LIE $\mathfrak{g}$–BUNCHES AND ISOTOPIC PAIRS

Definition 6.

A [6]. Let $\mathfrak{g}$ be a Lie algebra. A *Lie $\mathfrak{g}$–bunch* is a $\mathfrak{g}$–module V such that there exists a $\mathfrak{g}$–equivariant mapping $\mathfrak{g}\otimes\bigwedge^2(V)\mapsto V$, which defines a structure of Lie algebra in V when the first argument is fixed in an arbitrary way; we shall denote this mapping by $[\cdot,\cdot]_A$, $A\in\mathfrak{g}$.

B. A Lie $\mathfrak{g}$–bunch V is called *complete* if the Lie brackets defined by elements of $\mathfrak{g}$ are closed under the $\Diamond$ operation, i.e.

$$\forall A,B\in\mathfrak{g}\ \forall X\in V\ \exists C=A\underset{X}{\Diamond}B\in\mathfrak{g}:$$
$$[X,Y]_C=\tfrac{1}{2}([[X,Z]_A,Y]_B+[[X,Y]_A,Z]_B+[[Z,Y]_A,X]_B-(\alpha\longleftrightarrow\beta)).$$

Numerous examples of Lie $\mathfrak{g}$–bunches are collected in [9].

Proposition 2. *Each complete Lie $\mathfrak{g}$–bunch V may be enlarged to an isotopic pair $(V\oplus\mathbb{C},\mathfrak{g})$ with isocommutators*

$$[(X,\lambda\mathbf{1}),(Y,\mu\mathbf{1})]_A=([X,Y]_A+\lambda A(Y)-\mu A(X),0)$$
$$[A,B]_{(X,\lambda\mathbf{1})}=A\underset{X}{\Diamond}B+\lambda[A,B]$$

where $X,Y\in V$, $A,B\in\mathfrak{g}$; $A(X)$ is an image of an element $X\in V$ under the action of an element $A\in\mathfrak{g}$, $[A,B]$ is the commutator in $\mathfrak{g}$ and $[X,Y]_A$ is the isocommutator in V.

The proof of the proposition is straightforward.

APPENDIX B: ISOREPRESENTATIONS OF LIE ALGEBRAS VIA LIE SUPERALGEBRAS

Definition 7. Let $\mathfrak{g}$ be a Lie algebra. *An isorepresentation* (*Q–isorepresentation*) of $\mathfrak{g}$ in the linear space V is a pair (T,Q), $T\in Hom(\mathfrak{g},\mathrm{End}(V))$, $Q\in\mathrm{End}(V)$ such that $\forall X,Y\in\mathfrak{g}\ T(X)QT(Y)-T(Y)QT(X)=T([X,Y])$.

Note that a Lie algebra $\mathfrak{g}$ maybe considered as a complete Lie $\mathbb{C}$–bunch. The corresponding isotopic pair $(\mathbb{C},\mathfrak{g})$ will be denoted by $\mathcal{I}(\mathfrak{g})$. The isorepresentations of $\mathfrak{g}$ are just the representations of $\mathcal{I}(\mathfrak{g})$. The Lie superalgebra constructed from $\mathcal{I}(\mathfrak{g})$ is a standard Lie superalgebra associated with Lie algebra $\mathfrak{g}$ with even part

isomorphic to $\mathfrak{g}$, which acts in the odd part as in $\mathrm{ad}_{\mathfrak{g}}\oplus\mathbf{1}_{\mathfrak{g}}$ ($\mathrm{ad}_{\mathfrak{g}}$ is the adjoint $\mathfrak{g}$–module and $\mathbf{1}_{\mathfrak{g}}$ is the trivial one–dimensional $\mathfrak{g}$–module). The mapping from the symmetric square of the odd part to the even part is natural: $\mathrm{ad}_{\mathfrak{g}}\otimes\mathbf{1}_{\mathfrak{g}}\mapsto\mathfrak{g}$. It immediately gives a description of irreducible representations of $\mathfrak{g}$. First of all, let's consider a crucial example:

Example. Let (T_Q,Q) be an isorepresentation of $\mathfrak{g}$ in the space V. Then V admits two natural representation $T^{\pm}$ of the Lie algebra $\mathfrak{g}$: $\forall X\in\mathfrak{g}$ $T^+(X)=QT_Q(X)$ and $T^-(X)=T_Q(X)Q$, which are equivalent if Q is invertible.
Let T be a representation of $\mathfrak{g}$ in the space V and $Q\in\mathrm{GL}(V)$, then one may construct two equivalent isorepresentations $(T_Q^{\pm},Q)$ of $\mathfrak{g}$ in V: $\forall X\in\mathfrak{g}$ $T_Q^+(X)=Q^{-1}T(X)$ and $T_Q^-(X)=T(X)Q^{-1}$.

Second, one may restrict himself by split isorepresentations (i.e. split representations of $\mathcal{I}(\mathfrak{g})$).

Proposition 3. *Each finite dimensional irreducible split isorepresentation of $\mathfrak{g}$ may be realized in the linear space $V=V_1\oplus V_2$ ($V_1\simeq V_2\simeq V_\alpha$, where V_α carries an irreducible representation T_α of $\mathfrak{g}$) and Q is an isomorphism of V_1 onto V_2.*

Example (Two–dimensional irreducible isorepresentation of $\mathbb{C}$, cf.[9]). The operator Q have the form $\begin{pmatrix}0&0\\1&0\end{pmatrix}$, the generator of $\mathbb{R}$ is represented by $\begin{pmatrix}0&1\\0&0\end{pmatrix}$.

This example may be straightforwardly generalized on an arbitrary Lie algebra $\mathfrak{g}$. Namely, let $V=\mathrm{ad}_{\mathfrak{g}}\oplus\mathrm{ad}_{\mathfrak{g}}$, the operator Q has the form $\begin{pmatrix}0&0\\E&0\end{pmatrix}$, where E is a unit matrix and an element X of $\mathfrak{g}$ is represented by $\begin{pmatrix}0&\mathrm{ad}_{\mathfrak{g}}(X)\\0&0\end{pmatrix}$.

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